

AN INVARIANCE PRINCIPLE FOR THE LAW OF THE ITERATED LOGARITHM FOR SOME MARKOV CHAINS

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ABSTRACT. Strassen's invariance principle for additive functionals of Markov chains with spectral gap in the Wasserstein metric is proved.

1. INTRODUCTION

Suppose that (E, ρ) is a Polish space. By $\mathcal{B}(E)$ we denote the family of all Borel sets in E . By \mathcal{M}_1 we denote the space of all probability Borel measures on E . Let $\pi : E \times \mathcal{B}(E) \rightarrow [0, 1]$ be a transition probability on E . The Markov operator P is defined by $Pf(x) = \int_E f(y)\pi(x, dy)$ for every bounded Borel measurable function f on E . The same formula defines Pf for any Borel measurable functions $f \geq 0$ which need not be finite. Denote by $B_b(E)$ the set of all bounded Borel measurable functions equipped with the supremum norm and let $C_b(E)$ be its subset consisting of all bounded continuous functions.

Suppose that $(X_n)_{n \geq 0}$ is an E -valued Markov chain, given over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose transition is π and its initial distribution is equal to μ_0 . Denote by \mathbb{E} the expectation corresponding to \mathbb{P} . We shall denote by μP , the associated transfer operator describing the evolution of the law of X_n . To be precise, μP is defined by the formula $\int_E f(x)\mu P(dx) = \int_E Pf(x)\mu(dx)$ for any $f \in B_b(E)$ and $\mu \in \mathcal{M}_1$. To simplify the notation we shall write $\langle f, \mu \rangle$ instead of $\int_E f(y)\mu(dy)$.

Given a Lipschitz function $\psi : E \rightarrow \mathbb{R}$ we define

$$S_n(\psi) := \psi(X_0) + \psi(X_1) + \dots + \psi(X_n) \quad \text{for } n \geq 0.$$

Our aim is to find conditions under which $S_n(\psi)$ satisfies the law of the iterated logarithm (**LIL**). This natural question is raised when central limit theorems (**CLT**) are verified. Since 1986 when Kipnis and Varadhan [11] proved the central limit theorem for additive functionals of stationary reversible ergodic Markov chains, it has been a huge amount of reviving attempts to do this in various settings and under different conditions (see [15, 16]). A common factor of the mentioned results was that they were established with respect to the stationary probability law of the chain. In [5] Derriennic and Lin answered the question about the validity of the **CLT** with respect to the law of the Markov chain starting at some

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point x . Namely, they proved that the **CLT** holds for almost every x with respect to the invariant initial distribution (see also [6] and the references therein). On the other hand, Guivarc'h and Hardy [7] proved the **CLT** for a class of Markov chains associated with the transfer operator having spectral gap. Recently Komorowski and Walczuk studied Markov processes with the transfer operator having spectral gap in the Wasserstein metric and proved the **CLT** in the non-stationary case (see [12]). Other interesting results under similar assumptions were obtained by S. Kuksin and A. Shirikyan (see [13, 20]).

The **LIL** we study in this note was also considered in many papers. There are several results governing, for instance, the Harris recurrent chains [2, 3, 17]. Similarly the **CLT** results they are formulated mostly for stationary ergodic chains (see for instance [1, 4, 26, 18, 25]). In the case when one is able to find the solution to the Poisson equation $h = f + Ph$, the problem may be reduced to the martingale case [8] (see also [17]). But the **LIL** for martingales was carefully examined in many papers (see [10, 9, 22, 23]) and a lot of satisfactory results were obtained.

Our note is aimed at proving the **LIL** for Markov chains that satisfy the spectral gap property in the Wasserstein metric. It is worth mentioning here that many Markov chains satisfy this property, e.g. Markov chains associated with iterated function systems or stochastic differential equations disturbed with Poisson noise (see [14]).

Our result is based upon the **LIL** for martingales due to Heyde and Scott (see Theorem 1 in [9]).

2. ASSUMPTIONS AND AUXILIARY RESULTS

For every measure $\nu \in \mathcal{M}_1$ the law of the Markov chain $(X_n)_{n \geq 0}$ with transition probability π and initial distribution ν , is the probability measure \mathbb{P}_ν on $(E^\mathbb{N}, \mathcal{B}(E)^{\otimes \mathbb{N}})$ such that:

$$\mathbb{P}_\nu[X_{n+1} \in A | X_n = x] = \pi(x, A) \quad \text{and} \quad \mathbb{P}_\nu[X_0 \in A] = \nu(A),$$

where $x \in E$, $A \in \mathcal{B}(E)$. The expectation with respect to \mathbb{P}_ν is denoted by \mathbb{E}_ν . For $\nu = \delta_x$, the Dirac measure at $x \in E$, we write just \mathbb{P}_x and \mathbb{E}_x .

We will make the following assumption:

(H0) the Markov operator satisfies the Feller property, i.e. $P(C_b(E)) \subset C_b(E)$.

We shall denote by $\mathcal{M}_{1,1}$ the space of all probability measures possessing finite first moment, i.e. $\nu \in \mathcal{M}_{1,1}$ iff $\nu \in \mathcal{M}_1$ and $\int_E \rho(x_0, x) \nu(dx) < \infty$ for some (thus all) $x_0 \in E$. For abbreviation we shall write $\rho_{x_0}(x) = \rho(x_0, x)$. We assume that:

(H1) for any $\nu \in \mathcal{M}_{1,1}$ we have $P\nu \in \mathcal{M}_{1,1}$.

It may be proved that the space $\mathcal{M}_{1,1}$ is a complete metric space when equipped with the Wasserstein metric

$$d(\nu_1, \nu_2) = \sup\{|\langle f, \nu_1 \rangle - \langle f, \nu_2 \rangle| : f : E \rightarrow \mathbb{R}, \text{Lip } f \leq 1\}$$

for $\nu_1, \nu_2 \in \mathcal{M}_{1,1}$ and the convergence in the Wasserstein metric is equivalent to the weak convergence, see e.g. [24]. (Here $\text{Lip } f$ denotes the Lipschitz constant of f .) The main assumption made in our note says that the Markov operator P is contractive with respect to the Wasserstein metric, i.e.

(H2) there exist $\gamma \in (0, 1)$ and $c > 0$ such that

$$(2.1) \quad d(\mu P^n, \nu P^n) \leq c\gamma^n d(\mu, \nu), \quad \text{for } n \geq 1, \mu, \nu \in \mathcal{M}_{1,1}.$$

Let $\mu \in \mathcal{M}_{1,1}$. From now on we shall assume that the initial distribution of $(X_n)_{n \geq 0}$ is μ . Moreover,

(H3) there exists $x_0 \in E$ and $\delta > 0$ such that

$$(2.2) \quad \sup_{n \geq 0} \mathbb{E}_\mu \rho_{x_0}^{2+\delta}(X_n) < \infty.$$

It is easy to prove that under the assumptions (H0)–(H3) there exists a unique invariant (ergodic) measure $\mu_* \in \mathcal{M}_1$. In particular, $\mu_* \in \mathcal{M}_{1,1}$. The proof was given for Markov processes with continuous time in [12] but it still remains valid in discrete case.

Let $n_0 \geq 2$ be such that

$$\gamma_0 = c^2 \gamma^{n_0} < 1.$$

We start this part of the paper with a rather technical lemma.

Lemma 1. *Let $g_{n,k} : E^{2(k+n)} \rightarrow \mathbb{R}$ for arbitrary $k, n \geq 1$, be Lipschitz continuous in each variable with the same Lipschitz constant L . Then there exists constant \tilde{L} dependent only on L and such that the function*

$$(2.3) \quad \begin{aligned} H_{n,k}(x) &= \int_E \pi_1(x, dy_1) \int_E \pi_2(y_1, dy_2) \cdots \int_E \pi_{2(k+n)-1}(y_{2(k+n)-2}, dy_{2(k+n)-1}) \\ &\quad \times \int_E \pi_{2(k+n)}(y_{2(k+n)-1}, dy_{2(k+n)}) g_{n,k}(y_1, \dots, y_{2(k+n)}), \end{aligned}$$

where $\pi_l(y_{l-1}, dy_l) = \delta_{y_{l-1}} P^{k_l}(dy_l)$, $k_l \geq 1$ and additionally $k_l \geq n_0 - 1$ for all even l , is Lipschitzian with the Lipschitz constant \tilde{L} .

Proof. Define the functions $g_j : E^j \rightarrow \mathbb{R}$ by the formula

$$\begin{aligned} g_j(y_0, y_1, \dots, y_{j-1}) &= \int_E \pi_j(y_{j-1}, dy_j) \int_E \pi_{j+1}(y_j, dy_{j+1}) \times \cdots \\ &\quad \times \int_E \pi_{2(k+n)}(y_{2(k+n)-1}, dy_{2(k+n)}) g_{n,k}(y_1, \dots, y_{2(k+n)}) \quad \text{for } j = 1, \dots, 2(k+n). \end{aligned}$$

Let $\mathcal{L}_{j,l}$ for $j = 1, \dots, 2(k+n)$ and $l = 0, \dots, j-1$ denote the Lipschitz constant of g_j with respect to y_l . Then the Lipschitz constant of $H_{n,k}$ is equal to $\mathcal{L}_{1,0}$. It is obvious that $\mathcal{L}_{j,l} \leq L$ for $0 \leq l < j-1$, $j > 1$. To evaluate $\mathcal{L}_{j,j-1}$ fix y_0, y_1, \dots, y_{j-2} and $\tilde{y}_{j-1}, \hat{y}_{j-1}$. Then we have

$$\begin{aligned} &g_j(y_0, y_1, \dots, y_{j-2}, \hat{y}_{j-1}) - g_j(y_0, y_1, \dots, y_{j-2}, \tilde{y}_{j-1}) \\ &= \int_E \pi_j(\hat{y}_{j-1}, dy_j) g_{j+1}(y_0, y_1, \dots, \hat{y}_{j-1}, y_j) - \int_E \pi_j(\tilde{y}_{j-1}, dy_j) g_{j+1}(y_0, y_1, \dots, \tilde{y}_{j-1}, y_j) \\ &= \int_E \pi_j(\hat{y}_{j-1}, dy_j) (g_{j+1}(y_0, y_1, \dots, \hat{y}_{j-1}, y_j) - g_{j+1}(y_0, y_1, \dots, \tilde{y}_{j-1}, y_j)) \\ &\quad + \int_E \pi_j(\hat{y}_{j-1}, dy_j) g_{j+1}(y_0, y_1, \dots, \tilde{y}_{j-1}, y_j) - \int_E \pi_j(\tilde{y}_{j-1}, dy_j) g_{j+1}(y_0, y_1, \dots, \tilde{y}_{j-1}, y_j) \end{aligned}$$

and consequently

$$\begin{aligned} |g_j(y_0, y_1, \dots, \hat{y}_{j-1}) - g_j(y_0, y_1, \dots, \tilde{y}_{j-1})| &\leq \mathcal{L}_{j+1, j-1} \rho(\hat{y}_{j-1}, \tilde{y}_{j-1}) \int_E \pi_j(\hat{y}_{j-1}, dy_j) \\ &+ |\langle P^{k_j} \tilde{g}_{j+1}, \delta_{\hat{y}_j} \rangle - \langle P^{k_j} \tilde{g}_{j+1}, \delta_{\tilde{y}_j} \rangle| \leq L \rho(\hat{y}_{j-1}, \tilde{y}_{j-1}) + c_j \mathcal{L}_{j+1, j} \rho(\hat{y}_{j-1}, \tilde{y}_{j-1}), \end{aligned}$$

where $c_j = c\gamma$ if j odd, $c_j = c\gamma^{n_0-1}$ if j even and $\tilde{g}_{j+1}(\cdot) = g_{j+1}(y_0, y_1, \dots, y_{j-2}, \tilde{y}_{j-1}, \cdot)$. Hence we have

$$\mathcal{L}_{j, j-1} \leq L + c_j \mathcal{L}_{j+1, j} \quad \text{for } j = 1, \dots, 2(k+n) - 1.$$

Since $\mathcal{L}_{2(k+n), 2(k+n)-1} \leq L$, an easy computation shows that

$$\mathcal{L}_{1,0} \leq \frac{L(c\gamma + 1)}{1 - \gamma_0}.$$

This completes the proof. \square

3. THE LAW OF THE ITERATED LOGARITHM.

3.1. A martingale result. We start with recalling a classical result due to C.C. Heyde and D.J. Scott [9]. Let $\{S_n, \mathcal{F}_n : n \geq 0\}$ be a martingale on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and \mathcal{F}_n is the σ -field generated by S_1, S_2, \dots, S_n for $n > 0$. Let $S_0 = Z_0 = 0$ \mathbb{P} -a.s. and $S_n = \sum_{k=1}^n Z_k$ for $n \geq 1$. Further, let $s_n^2 = \mathbb{E}S_n^2 < \infty$.

We consider the metric space $(C, \tilde{\rho})$ of all real-valued continuous functions on $[0, 1]$ with

$$\tilde{\rho}(x, y) = \sup_{0 \leq t \leq 1} |x(t) - y(t)| \quad \text{for } x, y \in C.$$

Let K be the set of absolutely continuous functions $x \in C$ such that $x(0) = 0$ and $\int_0^1 (x'(t))^2 dt \leq 1$.

Define the real function g on $[0, \infty)$ by $g(s) = \sup\{n : s_n^2 \leq s\}$. We define a sequence of real random functions η_n on $[0, 1]$, for $n > g(e)$, by

$$\eta_n(t) = \frac{S_k + (s_n^2 t - s_k^2)(s_{k+1}^2 - s_k^2)^{-1} Z_{k+1}}{\sqrt{2s_n^2 \log \log s_n^2}}$$

if $s_k^2 \leq s_n^2 t \leq s_{k+1}^2$, $k = 1, \dots, n-1$ and

$$\eta_n(t) = 0 \quad \text{for } n \leq g(e).$$

Proposition 1. (Theorem 1 in [9]) *If $s_n^2 \rightarrow \infty$ and*

$$(3.1) \quad \sum_{n=1}^{\infty} s_n^{-4} \mathbb{E}[Z_n^4 \mathbf{1}_{\{|Z_n| < \gamma s_n\}}] < \infty \quad \text{for some } \gamma > 0,$$

$$(3.2) \quad \sum_{n=1}^{\infty} s_n^{-1} \mathbb{E}[|Z_n| \mathbf{1}_{\{|Z_n| \geq \epsilon s_n\}}] < \infty \quad \text{for all } \epsilon > 0,$$

$$(3.3) \quad s_n^{-2} \sum_{k=1}^n Z_k^2 \rightarrow 1 \quad \mathbb{P}\text{-a.s. as } n \rightarrow \infty$$

hold, then $\{\eta_n\}_{n \geq 1}$ is relatively compact in C and the set of its limit points coincides with K .

3.2. Application to Markov chains. Let $\psi : E \rightarrow \mathbb{R}$ be a Lipschitz function such that $\langle \psi, \mu_* \rangle = 0$, otherwise we could consider $\tilde{\psi} = \psi - \langle \psi, \mu_* \rangle$. Let $L > 0$ denote its Lipschitz constant. Let $(X_n)_{n \geq 0}$ be a Markov chain with the initial distribution μ satisfying conditions (H0)–(H3).

We have $\sum_{i=0}^{\infty} |P^i \psi(x)| = \sum_{i=0}^{\infty} |\langle \psi, \delta_x P^i \rangle - \langle \psi, \mu_* P^i \rangle| \leq cd(\delta_x, \mu_*) \sum_{i=0}^{\infty} \gamma^i < \infty$, by (H2). Thus we may define the function

$$\chi(x) := \sum_{i=0}^{\infty} P^i \psi(x) \quad \text{for } x \in E.$$

We easily check that χ is a Lipschitz function.

It is well known that

$$S_n = \chi(X_n) - \chi(X_0) + \sum_{i=0}^n \psi(X_i) \quad \text{for } n \geq 0$$

is a martingale on the space $(E^{\mathbb{N}}, \mathcal{B}(E)^{\otimes \mathbb{N}}, \mathbb{P}_\mu)$ with respect to the natural filtration and its square integrable martingale differences are of the form

$$Z_n = \chi(X_n) - \chi(X_{n-1}) + \psi(X_n) \quad \text{for } n \geq 1.$$

Observe that $\mathbb{E}_{\mu_*} Z_1^2 < \infty$. Indeed, we easily check that $x \rightarrow \mathbb{E}_x(Z_1^2 \wedge k)$ for any $k \geq 1$ is a bounded continuous function. Further, since $\mathbb{E}_{\mu P^n}(Z_1^2 \wedge k) = \int_E \mathbb{E}_x(Z_1^2 \wedge k) \mu P^n(dx) \rightarrow \mathbb{E}_{\mu_*}(Z_1^2 \wedge k)$ for any $k \geq 1$ as $n \rightarrow \infty$ and $\sup_{n \geq 0} \mathbb{E}_{\mu P^n}(Z_1^2) < \infty$, we obtain that $\mathbb{E}_{\mu_*}(Z_1^2) < \infty$.

Set

$$\sigma^2 := \mathbb{E}_{\mu_*} Z_1^2.$$

We have

$$(3.4) \quad \lim_{n \rightarrow \infty} \mathbb{E}_{\mu P^n} Z_1^2 = \lim_{n \rightarrow \infty} \mathbb{E}_\mu Z_n^2 = \sigma^2.$$

In fact, since χ and ψ are Lipschitzian, we have $\sup_{n \geq 1} \mathbb{E}_\mu |Z_n|^{2+\delta} < \infty$, by (H3). Further, observe that

$$\sup_{n \geq 1} \mathbb{E}_\mu (Z_n^2 \mathbf{1}_{\{|Z_n|^2 \geq k\}}) \leq k^{-\delta/2} \sup_{n \geq 1} \mathbb{E}_\mu |Z_n|^{2+\delta} \rightarrow 0$$

as $k \rightarrow \infty$. Therefore, condition (3.4) follows from the fact that $\mathbb{E}_{\mu P^n}(Z_1^2 \wedge k) \rightarrow \mathbb{E}_{\mu_*}(Z_1^2 \wedge k)$ as $n \rightarrow \infty$ for any $k \geq 1$. Finally, we obtain

$$\lim_{n \rightarrow \infty} \frac{s_n^2}{n} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}_\mu S_n^2}{n} = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \mathbb{E}_\mu Z_i^2}{n} = \sigma^2.$$

Lemma 2. *The square integrable martingale differences $(Z_n)_{n \geq 1}$ satisfy the following condition:*

$$(3.5) \quad \frac{1}{n} \sum_{l=1}^n Z_l^2 \rightarrow \sigma^2 \quad \mathbb{P}_\mu\text{-a.s. as } n \rightarrow \infty$$

and consequently if $\sigma^2 > 0$ condition (3.3) holds.

Proof. First observe that to finish the proof it is enough to show that for any $i \in \{1, \dots, n_0\}$ we have

$$\frac{1}{n} \sum_{l=1}^n Z_{i+ln_0}^2 \rightarrow \sigma^2 \quad \mathbb{P}_\mu\text{-a.s. as } n \rightarrow \infty.$$

If we show that both the functions

$$x \rightarrow \mathbb{E}_x(|\liminf_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) - \sigma^2| \wedge 1)$$

and

$$x \rightarrow \mathbb{E}_x(|\limsup_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) - \sigma^2| \wedge 1)$$

are continuous, we shall be done. Indeed, then we have

$$\begin{aligned} \mathbb{E}_\mu(|\liminf_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) - \sigma^2| \wedge 1) &= \int_E \mathbb{E}_x(|\liminf_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) - \sigma^2| \wedge 1) \mu(dx) \\ &= \int_E \mathbb{E}_x(|\liminf_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) - \sigma^2| \wedge 1) \mu P^{i+mn_0}(dx) \rightarrow \mathbb{E}_{\mu_*}(|\liminf_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) - \sigma^2| \wedge 1), \end{aligned}$$

as $m \rightarrow +\infty$, by the fact that μP^{i+mn_0} converges weakly to μ_* as $m \rightarrow +\infty$. On the other hand, from the Birkhoff individual ergodic theorem we have

$$\mathbb{E}_{\mu_*}(|\liminf_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) - \sigma^2| \wedge 1) = 0$$

and consequently

$$\mathbb{E}_\mu(|\liminf_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) - \sigma^2| \wedge 1) = 0,$$

which, in turn, gives

$$\liminf_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) = \sigma^2 \quad \mathbb{P}_\mu\text{-a.s.}$$

Analogously we may show that

$$\limsup_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) = \sigma^2 \quad \mathbb{P}_\mu\text{-a.s.}$$

The remainder of the proof is devoted to showing the continuity of the relevant functions. Again, we restrict to the first function, since the proof for the second one goes in almost the same manner.

Observe that

$$\begin{aligned} & \mathbb{E}_x(|\liminf_{n \rightarrow \infty} (1/n \sum_{l=1}^n Z_{i+ln_0}^2) - \sigma^2| \wedge 1) \\ &= \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \mathbb{E}_x \left(\left| \min \left\{ \frac{1}{n} \sum_{l=1}^n Z_{i+ln_0}^2 - \sigma^2, \dots, \frac{1}{n+k} \sum_{l=1}^{n+k} Z_{i+ln_0}^2 - \sigma^2 \right\} \right| \wedge 1 \right) \\ &= \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} H_{n,k}(x), \end{aligned}$$

where

$$\begin{aligned} H_{n,k}(x) &= \mathbb{E}_x(|\min\{1/n(\sum_{l=1}^n Z_{i+ln_0}^2 \wedge n(1+\sigma^2)) - \sigma^2, \dots, \\ &1/(n+k)(\sum_{l=1}^{n+k} Z_{i+ln_0}^2 \wedge (n+k)(1+\sigma^2) - \sigma^2)\}| \wedge 1) \\ &= \mathbb{E}_x(|\min\{1/n(\sum_{l=1}^n (\chi(X_{i+ln_0}) - \chi(X_{i-1+ln_0}) + \psi(X_{i+ln_0}))^2 \wedge n(1+\sigma^2)) - \sigma^2, \dots, \\ &1/(n+k)(\sum_{l=1}^{n+k} (\chi(X_{i+ln_0}) - \chi(X_{i-1+ln_0}) + \psi(X_{i+ln_0}))^2 \wedge (n+k)(1+\sigma^2) - \sigma^2)\}| \wedge 1). \end{aligned}$$

Set

$$\begin{aligned} g_{n,k}(y_1, \dots, y_{2(n+k)}) &= |\min\{1/n(\sum_{l=1}^n (\chi(y_{2l}) - \chi(y_{2l-1}) + \psi(y_{2l}))^2 \wedge n(1+\sigma^2)) - \sigma^2, \dots, \\ &1/(n+k)(\sum_{l=1}^{n+k} (\chi(y_{2l}) - \chi(y_{2l-1}) + \psi(y_{2l}))^2 \wedge (n+k)(1+\sigma^2) - \sigma^2)\}| \wedge 1 \end{aligned}$$

so that

$$H_{n,k}(x) = \mathbb{E}_x(g_{n,k}(X_{i+n_0-1}, X_{i+n_0}, X_{i+2n_0-1}, X_{i+2n_0}, \dots, X_{i+2(n+k)n_0-1}, X_{i+2(n+k)n_0})).$$

Observe that $H_{n,k}$ is given by formula (2.3). If we show that there exists L such that $g_{n,k}$ is Lipschitz continuous in each variable with the Lipschitz constant L (independent of n, k), then all $H_{n,k}$ are Lipschitz with the same Lipschitz constant \tilde{L} , by Lemma 1. Consequently $\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} H_{n,k}$ is Lipschitz and in particular continuous. Since minimum of any finite family of functions which are Lipschitz continuous in each variable with the Lipschitz constant L is Lipschitz continuous in each variable with the same Lipschitz

constant L , to finish the proof it is enough to observe that the function

$$(y_1, \dots, y_{2p}) \rightarrow 1/p \left(\sum_{l=1}^p (\chi(y_{2l}) - \chi(y_{2l-1}) + \psi(y_{2l}))^2 \wedge p(1 + \sigma^2) \right) - \sigma^2$$

is Lipschitz continuous in each variable with the Lipschitz constant L for fixed $L > 0$. On the other hand, each term in the above sum is Lipschitz continuous in each variable with the Lipschitz constant

$$(1/p)(\text{Lip } \chi + \text{Lip } \psi)2p(1 + \sigma^2) = 2(\text{Lip } \chi + \text{Lip } \psi)(1 + \sigma^2).$$

Observe that each variable appears in one term in the above sum. Hence $L \leq 2(\text{Lip } \chi + \text{Lip } \psi)(1 + \sigma^2)$, which finishes the proof. \square

Note that following the proof of our previous lemma we are able to show that the considered Markov chain satisfies the strong law of large numbers (**SLLN**). This result however directly follows from Theorem 2.1 in [19].

Lemma 3. *Let $\sigma^2 > 0$. Under the assumptions (H0)–(H3) the square integrable martingale differences $(Z_n)_{n \geq 1}$ satisfy conditions (3.1), (3.2).*

Proof. Since $\sup_{n \geq 1} \mathbb{E}_\mu |Z_n|^{2+\delta} < \infty$, δ is the constant given in (H3), we have

$$\sum_{n=1}^{\infty} s_n^{-4} \mathbb{E}_\mu [Z_n^4 \mathbf{1}_{\{|Z_n| < \gamma s_n\}}] \leq \sum_{n=1}^{\infty} s_n^{-4} \gamma^{2-\delta} s_n^{2-\delta} \mathbb{E}_\mu |Z_n|^{2+\delta} \leq \gamma^{2-\delta} \sup_{n \geq 1} \mathbb{E}_\mu |Z_n|^{2+\delta} \sum_{n=1}^{\infty} s_n^{-2-\delta}.$$

On the other hand, the condition $s_n^2/n \rightarrow \sigma^2$ as $n \rightarrow \infty$ gives $\sum_{n=1}^{\infty} s_n^{-2-\delta} < \infty$, which completes the proof of condition (3.1).

To show condition (3.2) observe that

$$\sum_{n=1}^{\infty} s_n^{-1} \mathbb{E}_\mu [|Z_n| \mathbf{1}_{\{|Z_n| \geq \epsilon s_n\}}] \leq \sum_{n=1}^{\infty} s_n^{-1} \mathbb{E}_\mu [|Z_n|^{2+\delta} / (\epsilon s_n)^{1+\delta}] \leq \epsilon^{-1-\delta} \sup_{n \geq 1} \mathbb{E}_\mu |Z_n|^{2+\delta} \sum_{n=1}^{\infty} s_n^{-2-\delta} < \infty.$$

The proof is complete. \square

3.3. The Law of Iterated Logarithm for Markov chains.

Theorem 1. *Let $(X_n)_{n \geq 0}$ be a Markov chain with an initial distribution μ satisfying conditions (H0)–(H3). If ψ is a Lipschitz function with $\langle \psi, \mu_* \rangle = 0$ and $\sigma^2 > 0$, then \mathbb{P}_μ -a.s. the sequence*

$$\theta_n(t) = \frac{\sum_{i=1}^k \psi(X_i) + (nt - k)\psi(X_{k+1})}{\sigma \sqrt{2n \log \log n}}$$

if $k \leq nt \leq k+1$, $k = 1, \dots, n-1$ for $t > 0$, $n > e$ and $\theta_n(t) = 0$ otherwise is relatively compact in C and the set of its limit points coincides with K .

Proof. First observe that since $s_n^2/n \rightarrow \sigma^2 > 0$ as $n \rightarrow \infty$ we have

$$\frac{\sqrt{2s_n^2 \log \log s_n^2}}{\sigma \sqrt{2n \log \log n}} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Consequently, from Lemmas 2 and 3 it follows that the sequence

$$\eta_n(t) = \frac{S_k + (s_n^2 t - s_k^2)(s_{k+1}^2 - s_k^2)^{-1} Z_{k+1}}{\sigma \sqrt{2n \log \log n}}$$

if $s_k^2 \leq s_n^2 t \leq s_{k+1}^2$, $k = 1, \dots, n-1$ for $t > 0, n > e$ and $\eta_n(t) = 0$ otherwise is relatively compact in C and the set of its limit points coincides with K , due to Heyde and Scott [9]. Let $t \in (0, 1]$ and $n \geq 1$. Observe that if $k \leq nt \leq k+1$, then

$$\frac{k\sigma^2}{s_k^2} s_k^2 \leq \frac{n\sigma^2}{s_n^2} t s_n^2 \leq \frac{(k+1)\sigma^2}{s_{k+1}^2} s_{k+1}^2.$$

Set

$$\hat{\eta}_n(t) = \frac{S_k + (nt - k)Z_{k+1}}{\sigma \sqrt{2n \log \log n}},$$

where $k \geq 1$ such that $k \leq nt \leq k+1$. Since $(n\sigma^2)/s_n^2 \rightarrow 1$ as $n \rightarrow \infty$ for any $\varepsilon > 0$ holds

$$(1 - \varepsilon)s_k^2 \leq (1 + \varepsilon)s_n^2 t \leq (1 + \varepsilon)^2(1 - \varepsilon)^{-1}s_{k+1}^2$$

for all n large enough. Hence there is $t_* \in [t(1 - \varepsilon)(1 + \varepsilon)^{-1}, t(1 + \varepsilon)(1 - \varepsilon)^{-1}]$ such that $s_k^2 \leq s_n^2 t_* \leq s_{k+1}^2$. On the other hand, the diameter of the interval $[s_k^2/s_n^2, s_{k+1}^2/s_n^2]$ for a fixed $k = 1, \dots, n-1$ converges to 0 as $n \rightarrow \infty$. Consequently, for any $t > 0$ and $n > e$ there exists $t_n > 0$ such that $\hat{\eta}_n(t) = \eta_n(t_n)$ and $t_n \rightarrow t$ as $n \rightarrow \infty$. Since the sequence $(\eta_n(t))_{n>e}$ is relatively compact in C and the set of its limit points coincides with K , the sequence $(\hat{\eta}_n(t))_{n>e}$ is also relatively compact and has the same set of limits points.

Fix $\varepsilon > 0$. Define the sets

$$A_n = \left\{ \omega \in \Omega : \frac{|S_n - \sum_{i=1}^n \psi(X_i)|}{\sqrt{n}} \geq \varepsilon/2 \right\} \cup \left\{ \omega \in \Omega : \frac{|Z_n - \psi(X_n)|}{\sqrt{n}} \geq \varepsilon/2 \right\} \quad \text{for } n \geq 1.$$

Now we are going to show that $\sum_{n=1}^{\infty} \mathbb{P}_{\mu}(A_n) < \infty$. Indeed, keeping in mind that χ is Lipschitzian, by the Chebyshev inequality we obtain

$$\begin{aligned} \mathbb{P}_{\mu} \left(\left\{ \omega \in \Omega : \frac{|S_n - \sum_{i=1}^n \psi(X_i)|}{\sqrt{n}} \geq \varepsilon/2 \right\} \right) &= \mathbb{P}_{\mu} \left(\left\{ \omega \in \Omega : \frac{|\chi(X_n) - \chi(X_0)|}{\sqrt{n}} \geq \varepsilon/2 \right\} \right) \\ &\leq c_0 \frac{\mathbb{E}(\rho_{x_0}(X_n))^{2+\delta} + \mathbb{E}(\rho_{x_0}(X_0))^{2+\delta}}{n^{1+\delta/2}} \leq \frac{c}{n^{1+\delta/2}}, \end{aligned}$$

by (H3) for some constant $c > 0$ independent of n .

Analogously, we may check that there exists a positive constant C (independent of n) such that

$$\begin{aligned} \mathbb{P}_{\mu} \left(\left\{ \omega \in \Omega : \frac{|Z_n - \psi(X_n)|}{\sqrt{n}} \geq \varepsilon/2 \right\} \right) &= \mathbb{P}_{\mu} \left(\left\{ \omega \in \Omega : \frac{|\chi(X_n) - \chi(X_{n-1})|}{\sqrt{n}} \geq \varepsilon/2 \right\} \right) \\ &\leq \frac{C}{n^{1+\delta/2}}, \end{aligned}$$

by (H3) and the Lipschitz property of the function χ . Thus the series $\sum_{n=1}^{\infty} \mathbb{P}_{\mu}(A_n)$ is convergent.

Finally, from the Borel–Cantelli lemma it follows that \mathbb{P}_μ -a.s.

$$\limsup_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} \left| \frac{S_k + (nt - k)Z_{k+1}}{\sigma \sqrt{2n \log \log n}} - \frac{\sum_{i=1}^k \psi(X_i) + (nt - k)\psi(X_{k+1})}{\sigma \sqrt{2n \log \log n}} \right| < \varepsilon,$$

where $k \leq nt \leq k + 1$. Since $\varepsilon > 0$ was arbitrary, the proof is complete. \square

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